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The Banach Ball Property for the State Space of Order Unit Spaces

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ABSTRACT

In [1], Alfsen and Shultz have proved that the state space $S(A)$ of a JB-algebra A has the Hilbert ball property: for each pair ρ, σ of extreme points of $S(A)$, the face generated by ρ and σ is a norm-exposed face affinely isomorphic to the closed unit ball in some Hilbert space ([1], Corollary 3.12). Conversely, they have proved that the order unit space A being in spectral duality with its predual space is a $JB-$ algebra if the state space of A has the Hilbert ball property ([1], Theorem 7.2). We define the Banach ball property for the state space of order unit spaces and study order unit spaces which have this property. Thus, we describe some class of order unit spaces, geometry of which is similar to geometry of JB-algebras. Due to this fact we can develop the theory of order unit spaces like the theory of Jordan Banach spaces and obtain new results.

Keywords: order unit space, homogeneous factor, state space, trace, Banach ball property, generalized spin factor, JB-algebra.

INTRODUCTION

Let A be a real ordered linear space. We denote by A^+ the set of positive elements of A. An element $e \in A^+$ is called *order unit* if for every $a \in A$ there exists a number $\lambda \in R^+$ such, that $-\mathbf{e} \leq a \leq \lambda\mathbf{e}$. If the order is Archimedean, then the mapping $a \to \|a\| = \inf\{\lambda > 0 : -\lambda\mathbf{e} \leq a \leq \lambda\mathbf{e}\}$ is a norm. If A is a Banach space with respect to this norm, we say that (A, e) is an *order unit space with order unit **e*** [2].

A *base - norm space* is a positively generated partially ordered normed vector space V with distinguished base K of the positive cone V^+ such that the closed unit ball coincides with conv$(\boldsymbol{K} \cup -\boldsymbol{K})$.

Let $(A, \mathbf{e})$ be an order unit space which is the dual of a base-norm space (V,K) and $V^* = A.$ Then one can identify A with the space $A^b(K)$ of all bounded affine functions on K. A face F of K is called *norm exposed* if there exists a norm-closed supporting hyper-plane which meets K exactly in F.

Positive projections $P,\ Q$ on A are called *quasicomplementary* (q.c.) if

$$ker^+P = im^+Q, \quad im^+K = ker^+Q.$$

A w*-continuous positive projection $P: A \to A$ with $\|P\| \le 1$ is said to be a P*-projection* if there exists an unique w*-continuous positive projection $P': A \to A$ with $\|P'\| \le 1$ such that $P,\ P'$ are q.c. and the dual projections $P^*,\ P'^*$ are also q.c.

If A is a JBW-algebra and V is its predual, then the P-projection is exactly the map $U_p : a \to \{pap\}$ where p is an idempotent and $\{pap\}$ denotes the Jordan triple product. In particular, $U_p a = pap$ with the self-adjoint projection p if A is the self-adjoint part of an von Neumann algebra.

Every P-projection P is associated with the projective unit $u_p = P\mathbf{e}$ and *projective face*

$$F_p = K \cap imP^* = \{\rho \in K \,|< P\mathbf{e}, p >= 1\}.$$

We denote the sets of all $P-$projections on $A,$ projective units in $A,$ and projective faces of K by $\mathbf{P},\ U$, and $F,$ respectively.

Two positive elements $a,\ b$ in A are called *orthogonal* if there exists a projective face F_P of K such that $a = 0$ on F_P and $b = 0$ on F^*. We shall write this fact briefly as $a \perp b.$

Spaces $(A, \mathbf{e})$ and (V,K) are *in the spectral duality* if

(i) Every norm-exposed face of K is projective.
(ii) Every $a \in A$ admits a unique decomposition $a = a^+ - a^-$ such that $a^+, a^- \in A^+$ and $a^+ \perp a^-$.

When A and V are in the spectral duality, the sets **P**, U, F are complete orthomodular logics.

We recall (see [1]) that a $P-$projection R is *central* if $R+R'=I$ where I is the identity mapping. We say that an order unit space A is a factor (under the given duality with V) if it contains no central $P-$projection except O and I. We say that the projective unit $u \in U$ is *an atom* if u is a minimal (non-zero) element of the logic U. A factor A has type I if it contains an atom. It follows from [2] that if u is an atom, then the corresponding projective face consists of exactly one (extreme) point, further we shall denote this point by $\hat{u}$.

An element $u \in U$ is called *finite* if it is the l.u.b. of a finite number of atoms. The minimal number of atoms, whose l.u.b. is u, is called *the dimension* of u. We say that a factor A has type I_n if the dimension of $\mathbf{e}$ is equal to n. An order unit space A is called *a homogeneous factor of type* I_n if A is a factor of type I_n and $\mathbf{e}$ is l.u.b. of exactly n orthogonal atoms.

Definition 1. We say that a set K (base V) is *smooth* if an every extreme point of K has only one supporting hyper plane. A set K is said to be *strictly convex* if each open line segment in K doesn't intersect with the boundary of K. In other words, any face K has the form $\{\sigma\}$ where σ is an extreme point of K.

Definition 2. We say that a convex set K *has the Banach ball property* if for every pair ρ, σ of extreme points of K the minimal face $FACE(\{\rho,\sigma\})$ generated by ρ and σ is a norm-exposed face, which is a smooth and strictly convex set, and it is affinely isomorphic to a closed unit ball in some Banach space.

Note that if two extreme points ρ, σ coincide, then $FASE(\{\rho,\sigma\})=\{\rho\}$; so if K has the Banach ball property, then every extreme point of K is norm-exposed.

Example 1. Let E be a Banach space. Put $A_s=\mathbf{R}+E^*$. Then A_s is an order unit space with respect to the order: $a=\alpha+x\geq 0 \Leftrightarrow^{def} \alpha\geq\|x\|$ and norm: $\|a\|=|\alpha|+\|x\|$ for every $a\in A_s$. In this case, A_s is an order unit space of type I_2 and A_s is a factor, therefore its state space K coincides with a shift of a closed unit ball E.

If the state space $S(A)$ of A (the unit ball of E) is a smooth and strictly convex set, then A has the Banach ball property. In the case where E is a Hilbert space, A is called *a spin-factor*. Therefore this space is called *a generalized spin factor* [3].

So, a factor A_s of type I_2 is a generalized spin factor if and only if its state space has the Banach ball property.

Example 2. Let $p>1$ be an arbitrary fixed number. We call a real symmetric 2×2 matrix $T=\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ *p-positive definite* if

$$\begin{cases} a+c+\left(|2b|^p+|a-c|^p\right)^{\frac{1}{p}}\geq 0, \\ a+c-\left(|2b|^p+|a-c|^p\right)^{\frac{1}{p}}\geq 0. \end{cases}$$

We shall denote this by $T\geq_p \theta$ where $\theta=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This is a new definition for positive definiteness of 2×2 symmetric matrices which generalizes the usual definition of positive definiteness for a matrix.

For $p=2$ these inequalities are equivalent to $a+c\geq 0$ and $ac\geq b^2$. It means that $T\geq 0$ in the usual sense.

We denote the set of p-positive defined symmetric 2×2 matrices as M_p^+. As it was shown in [4], M_p^+ is an order unit space of type I_2, and projective units have the form $\frac{1}{2}\begin{pmatrix} 1+t & t' \\ t' & 1-t \end{pmatrix}$ where the numbers t and t' satisfy the condition $|t|^p+|t'|^p=1$. Therefore the state space M_P^+ is isomorphic to the smooth, strongly convex set $K=\{(x;y)\,|\,|x|^q+|y|^q\leq 1\}$ in R^2, $\frac{1}{p}+\frac{1}{q}=1$. Hence, the space M_p^+ has the Banach ball property.

Example 3. Let A_s be the order unit space from the Example 1. Then A_s is not an algebra and has type I_2. Consider the projective tensor product $A_s\otimes A_s$. It is clear that the last space also is not an algebra and has type I_4.

Remark 1. At present, there not any examples of order unit spaces which are not algebras and have type I_n, $n \geq 3$, n – odd.
In the monograph [5], Day gives several equivalent conditions on smoothness and strong convexity for the unit ball of Banach spaces. For example, a convex set is smooth and strongly convex if and only if its intersection with a plane, i.e. a two-dimensional subspace, is a smooth and strongly convex set. This implies that the logic F does not contain any five-element sublogic. Therefore the following results are valid.

Theorem 1. If a factor A of type I_n has the Banach ball property, then A is a homogeneous factor.

Theorem 2. If an order unit space has the Banach ball property, then it is reflexive.

Let A be a factor of type I_n, (V, K) be a base-norm space. We suppose that A and V are in the spectral duality and $V^* = A$.

Definition 3. A linear function τ on A is said to be a *trace* if $\tau(Ra + R'a) = \tau(a)$ for every $a \in A$ and $R \in \mathbf{P}$.

Theorem 3. The state space of A has the Banach ball property if and only if there exists a trace on A.

Proof. *Necessity.* Let K be a state space of A with the Banach ball property. So, A is a homogeneous factor by Theorem 1. For atoms $u_0, \vartheta_0 \in U$ we denote $h = u_0 \vee \vartheta_0$, $h = He$, $\in \mathbf{P}$. Then, according to [3], im $H = H(A)$ is a homogeneous factor of type I_2 with the state space $K_0 = imH^* \cap K$. Let $\hat{u}_0, \hat{\vartheta}_0$ be the extreme points of K corresponding to u_0 and ϑ_0, respectively. Since K has the Banach ball property, the face $FACE(\{\hat{u}_0, \hat{\vartheta}_0\}) = K_0$ is isomorphic to the smooth and strictly convex unit ball of some Banach space and there exists a trace τ on $H(A)$ ([6], Lemma 3). From [6] it follows that $\tau(u) = \tau(\vartheta)$ for all atoms u and ϑ from A.

Let $\mathbf{e} = \sum_{i=1}^{n} u_i$, where $u_i \perp u_j$ ($i \neq j$), be the orthogonal decomposition of the unit element $\mathbf{e}$ into atoms. We denote by $\hat{u}_i$ the extreme point of K

corresponding to u_i. Set also $\tau = \frac{1}{n}\sum_{i=1}^{n} \hat{u}_i$. Then we have $\tau \in K$. Any P-projection R is the sum of orthogonal one-dimensional P-projections R_i corresponding to u_i from the k-dimensional space $R(A)$. Therefore $\tau(R_i x) = \frac{1}{n}\hat{u}_i(x)$ and $R(x) = \frac{1}{n}\sum_{i=1}^{k} \hat{u}_i(x)$. It follows that τ is a trace and it does not depend on choice of decomposition of $\mathbf{e}$ into orthogonal atoms.

Sufficiency. Let there exist a trace on A. Then for every subfactor $A_1 \subset A$ such that $A_1 = R(A)$, $R \in \mathbf{P}$, there exists a trace. Let ρ, σ be a pair of extreme points of K. We denote by u and ϑ the corresponding minimal projective units (atoms) for ρ and σ, respectively. Put $He = h = u \vee \vartheta$. Then we have from [6] that $(H(A), h)$ is a generalized spin factor with the state space $FACE(\{\rho, \sigma\})$. □

It appears, we can pick out generalized spin factors among homogeneous factors of type I_2.

Corollary 1. Let A be a homogeneous factor of type I_2. A is the generalized spin factor if and only if A has a trace.

Let A be on order unit space, and let τ be a trace on A. For every $a \in A$ we put $\|a\|_1 = \tau(|a|)$, where $|a|$ is the module of the element a [1]. The map $\|\cdot\|_1 : A \to R$ is a norm on A if the trace τ is faithful. Finally, we denote by $L_1(A)$ the completion of A with respect to the L_1-norm.

Theorem 4. Let A be a homogeneous factor of type I_n, and let $V^* = A$, τ be a trace on A. The spaces $L_1(A)$ and V are order and isometrically isomorphic if and only if A is a JBW - factor of type I_n.

Proof. In the case when A is a JBW - factor with a trace, order and isometric isomorphism between spaces $L_1(A)$ and V is proved in [7].

Conversely, let $L_1(A) \cong V$ and u, ϑ be some pair atoms in A. We denote $H\mathbf{e} = h = u \vee \vartheta$. Then $(H(A), h)$ is a homogeneous factor of type I_2 with the state space $K_0 = K \cap imH^* = FACE\left(\{\hat{u}, \hat{\vartheta}\}\right)$. Since $L_1(A) \cong V$, imply

$L_1(H(A)) \cong V_0 = H^*(V)$, then K_0 is the Hilbert ball. Therefore, K possesses the Hilbert ball property, since u and ϑ are arbitrary. By [1, Theorem 7.2] we have that A is a JBW-factor. □

Remark 2. In [6] theorem 4 is proved on the other methods.

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